On the convergence of the sequence $|\cos(n)|^{n^\alpha}$ for $\alpha > 0$.

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Disclaimer: These notes have not been peer-reviewed and polished. While I have used genAI to check and correct the results, there may still be mistakes present. Feel free to get in touch if you find an error.

1 Introduction

The questions whether sequence $|\cos(n)|^{n^\alpha}$ for $\alpha > 0$ converges or diverges is an interesting test on mathematical intuition. It is is easy to show that for $n > 0$, $\cos(n)$ lies within the open interval $(0, 1)$, so taking powers of it could drive the sequence to zero. However, as n becomes larger, it is able to approximate multiples of 2π arbitrarily well, hence $\cos(n)$ will get arbitrarily close to ± 1. This convergence questions therefore relates to how well large integers can approximate multiples of π, or equivalently, how well rational numbers can approximate π. These notes relates the converge of the sequence to the irrationality exponent of π. To be more specific, the sequence converges precisely when $\alpha > 2$, if and only if, the irrationality exponent of π is 2. The later property being one of the many open mathematical problems that have been allegedly solved by OpenAI in a recent manuscript dump. A weaker rule, in case the irrationality exponent is larger than 2 is provided in Proposition 1.

2 Proofs

We prove the divergence and convergence cases separately before combining them into a single proposition.

Lemma 1 (Divergence)**.** *Assume that* $\left|\pi - \frac{p}{q}\right| < q^{-\mu}$ *is satisfied by an infinite number of coprime integer pairs* (p, q) *for some* $\mu > 1$*. Then the sequence* $a_n = |\cos(n)|^{n^\alpha}$ *diverges for* $0 < \alpha \leq 2(\mu - 1)$*.*

Proof. Let (p_i, q_i) be a sequence of solutions to the inequality such that $p_i \to \infty$. Multiplying the inequality by q_i yields $|\pi q_i - p_i| < q_i^{1-\mu}$, i.e., we can write $p_i = \pi q_i + \epsilon_i$ with $|\epsilon_i| < q_i^{1-\mu}$. Because $\mu > 1$, we have $|\epsilon_i| \to 0$, hence there exists $c > 0$ such that for sufficiently large i we can bound

$$|\cos(p_i)| = |\cos(\pi q_i + \epsilon_i)| = |\cos(\epsilon_i)| \geq \exp(-c\epsilon_i^2) \geq \exp(-c\pi^2 q_i^{2-2\mu}).$$

Because $\frac{p_i}{q_i} \to \pi$, for sufficiently large i we have $q_i \geq Cp_i$, thereby

$$|\cos(p_i)| \geq \exp(-c\epsilon_i^2) \geq \exp(-cC^{2-2\mu} p_i^{2-2\mu}).$$

Taking both sides to the p_i^α-th power results in

$$|\cos(p_i)|^{p_i^\alpha} \geq \exp(-cC^{2-2\mu} p_i^{2-2\mu+\alpha}).$$

This lower-bound converges to 1 as $p_i \to \infty$ precisely when $0 < \alpha < 2(\mu - 1)$. Furthermore, the lower-bound is a non-zero constant when $\alpha = 2(\mu - 1)$, in which case the subsequence a_{p_i} is bounded away from zero.

To show divergence, we construct another sequence r_i such that $a_{r_i} \to 0$. In particular, we can take any increasing sequence r_i such that modulo 2π it lies in the interval $(\frac{\pi}{3}, \frac{2\pi}{3})$. Such sequence exists due to π being irrational and satisfies $|\cos(r_i)| \leq \frac{1}{2}$, hence $|\cos(r_i)|^{r_i^\alpha} \to 0$. Consequently, the subsequence a_{r_i} converges to zero, wheres a_{p_i} does not. Hence a_n diverges. □

Lemma 2 (Convergence)**.** *Assume that* $\left|\pi - \frac{p}{q}\right| < q^{-\mu}$ *is satisfied by at most a finite number of coprime integer pairs* (p, q) *for some* $\mu > 1$*. Then the sequence* $a_n = |\cos(n)|^{n^\alpha}$ *converges to zero for* $\alpha > 2(\mu - 1)$*.*

Proof. For sufficiently large i, define q_i be the nearest integer to $\frac{i}{\pi}$ and take ϵ_i such that $i = \pi q_i + \epsilon_i$. Because the pair (i, q_i) is not necessarily coprime, we let $P_i, Q_i \in \mathbb{Z}$ such that $\frac{i}{q_i} = \frac{P_i}{Q_i}$ and $\gcd(P_i, Q_i) = 1$. Then by construction the pair (P_i, Q_i) is coprime and $q_i \geq Q_i$. By assumption, there are only finitely many coprime pairs (P, Q) that satisfy $\left|\pi - \frac{P}{Q}\right| < Q^{-\mu}$. We can also rule out that the same coprime pair (P, Q) occurs infinitely often in the sequence (P_i, Q_i), because if it was the case, then $\frac{P}{Q}$ must coincide with the irrational limit $\frac{i}{q_i} \to \pi$, which contradicts $\frac{P}{Q}$ being rational. Thus, for sufficiently large i we obtain

$$|\pi - \tfrac{i}{q_i}| = |\pi - \tfrac{P_i}{Q_i}| \geq Q_i^{-\mu} \geq q_i^{-\mu},$$

or equivalently $|\epsilon_i| \geq q_i^{1-\mu}$. Furthermore, by construction $|\epsilon_i| \leq \frac{\pi}{2}$, hence there exists a constant $C > 0$ such that

$$|\cos(i)| = |\cos(\pi q_i + \epsilon_i)| = |\cos(\epsilon_i)| \leq \exp(-C\epsilon_i^2) \leq \exp(-Cq_i^{2-2\mu}).$$

Because $\frac{i}{q_i} \to \pi$, for sufficiently large i we have $i \geq cq_i$, thereby

$$|\cos(i)| \leq \exp(-Cq_i^{2-2\mu}) \leq \exp(-Cc^{2\mu-2}i^{2-2\mu}).$$

Taking both sides to the i^α-th power results in

$$|\cos(i)|^{i^\alpha} \leq \exp(-Cc^{2-2\mu}i^{2-2\mu+\alpha}).$$

This upper bound converges to zero when $\alpha > 2(\mu - 1)$. Hence, the sequence a_n converges when $\alpha > 2(\mu - 1)$. □

Definition 1. *The irrationality exponent $\mu(\beta)$ of a real number $\beta \in \mathbb{R}$ is the supremum of all μ such that $0 < |\beta - \frac{p}{q}| < q^{-\mu}$ is satisfied by an infinite number of coprime integer pairs (p, q). In particular, the inequality has an infinite number of coprime solutions for $\mu < \mu(\beta)$, and at most finitely many such pairs when $\mu > \mu(\beta)$.*

By Dirichlet's approximation theorem, for every irrational number β, its irrationality exponent satisfies $\mu(\beta) \geq 2$. For π, $\mu(\pi) = 2$ has been conjectured for a long time, and a proof of this equality has been claimed by OpenAI with a Lean proof [3], but has yet to be verified. The current best published upper bound is $\mu(\pi) \leq 7.1032\ldots$ from [4].

Proposition 1. *Let $\mu(\pi)$ be the irrationality exponent of π and define the sequence $a_n = |\cos(n)|^{n^\alpha}$ for all $n \in \mathbb{N}$. The sequence a_n converges if $\alpha > 2(\mu(\pi) - 1)$ and diverges if $0 < \alpha < 2(\mu(\pi) - 1)$ or $\alpha = 2$.*

Proof. Combine Lemmas 1 and 2 for the relations between α and $\mu(\pi)$; for the case $\alpha = 2$ use Dirichlet's approximation theorem with Lemma 1. □

If the OpenAI proof in [3] is correct, then this implies the following easy corollary.

Corollary 1. *The sequence $a_n = |\cos(n)|^{n^\alpha}$ converges if and only if $\alpha > 2$.*

This would solve the convergence question for all $\alpha > 0$. If it turns out that $\mu(\pi) > 2$, then the only case not settled by Proposition 1 is $\alpha = 2(\mu(\pi) - 1)$.

Acknowledgments

I originally asked about the limit $\lim_{n\to\infty} \cos^n(n)$ in a Quora post [1] around 2018. I do not recall how I originally came up with the problem, only that I was unable to solve it. The only correct answer on the Quora post [1] is by user Christian Schmidt, to whom I am grateful, but this answer is no longer visible due to Quora's policy on hyperlinks. Quora user Senia Sheydvasser commented an argument for the existence of a function $f(n)$ such that $\lim_{n\to\infty} \cos^{f(n)}(n) = 0$ without explicit construction. In 2023, a similar question about $\lim_{n\to\infty} \cos^{(n^2)}(n)$ had been asked on Stack Exchange [2]. This manuscript started when recalling the problem I stated in the Quora post after reading about OpenAI having allegedly proven that the irrationality exponent of π equals 2.

References

[1] What is the limit $\lim_{n\to\infty} \cos^n(n)$? https://www.quora.com/What-is-the-limit-displaystyle-lim_-n-to-infty-cos-n-n. Quora, accessed November 4, 2026.

[2] What is the limit $\lim_{n\to\infty} \cos(n)^{n^2}$? https://math.stackexchange.com/questions/4819898/what-is-the-limit-lim-n-to-infty-cosnn2. Question posted December 4, 2023, accessed November 4, 2026.

[3] OpenAI. The irrationality exponent of π is 2, September 2026. OpenAI Math Release preprint.

[4] Doron Zeilberger and Wadim Zudilin. The irrationality measure of π is at most 7.103205334137.... *Moscow Journal of Combinatorics and Number Theory*, 9(4):407–419, 2020.